Research Statement — Anirban Bhaduri

1. Introduction

My research interests lie in derived categories in the context of algebraic geometry, representation theory of rings and algebras, and related fields. Currently, my research focuses on derived categories of non-commutative algebras, schemes, and stacks using tools such as generators, thick subcategories, and invariants like generation time and Rouquier dimension. Broadly, I am interested in studying spaces such as non-commutative schemes arise from derived categories of modules over finite-dimensional k-algebras(or differential graded algebras) and its applications to areas like non-commutative resolution of singularities, homological mirror symmetry and even in string theory. Non-commutative algebraic geometry is extensively discussed in the works of [1, 23, 28].

Derived categories, rooted in the work of Grothendieck and Verdier, have become central in algebraic geometry by encoding information about varieties, schemes, and stacks via coherent sheaves. For example, the Bondal–Orlov reconstruction theorem [8] shows that if $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ for smooth irreducible projective varieties X and Y, then $X \cong Y$. The study of derived categories is also significant since it provides a connection between geometric objects and algebraic ones. At a certain level, one can study a variety by associating an algebra and working with the derived category of modules over the algebra. A theorem proved by both Baer [2] and Bondal [7] says that if we have certain special object (called *tilting objects*) T in the abelian category $\mathrm{coh}(X)$, where X is a smooth projective variety, then $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{mod}}(A)$, where A is a finite-dimensional algebra. The corresponding abelian categories are not equivalent and thus at the level of derived categories we find a connection between areas of algebraic geometry, representation theory, and non-commutative algebra. This association can be understood with the following example which is widely known in literature:

Example 1.1. Let $\mathbb{P}^1$ be the projective line over an algebraically closed field k. The category $\mathrm{coh}(\mathbb{P}^1)$ is an hereditary abelian category whose objects are direct sums of torsion and torsion-free. For the bounded derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1)$, we have

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{mod}}(kQ)$$

where kQ is a finite-dimensional $k-$algebra whose elements are the paths of the graph called Kronecker quiver given by

$$\bullet \rightrightarrows \bullet$$

This example has been explained in great detail in [20], Chapter 5. Note that $\mathrm{coh}(\mathbb{P}^1)$ is not equivalent to $\mathrm{mod}\,(kQ)$. Here, the *tilting object* T is given by $\mathcal{O} \oplus \mathcal{O}(1)$, where $\mathcal{O}$ is the structure sheaf. The functor

$$\mathrm{Hom}(T, -) : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{mod}}(kQ)$$

induces the triangle equivalence. Here, $T = \mathcal{O} \oplus \mathcal{O}(1)$ is called a *strong generator* which has *generation time* one. In [22], Proposition 15, it is shown that the set of all possible generation times, called the Orlov spectrum, for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1)$ is $\{1, 2\}$. Since 1 is the minimum generation time, it is called the *Rouquier dimension* of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1)$. This can be found in [29]. The category also has a *semiorthogonal decomposition*: $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1) \cong \langle\langle\mathcal{O}\rangle, \langle\mathcal{O}(1)\rangle\rangle$.

2. Derived Categories and Dimensions

During my Ph.D., my research focused on understanding bounded derived categories of algebras and stacks through semiorthogonal decompositions, generators, and invariants such as generation time, dimensions, and Orlov spectra. The bounded derived category $\mathrm{D^b}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is a triangulated category. For a triangulated category $\mathcal{T}$, a *classical generator* is an object G from which all others can be obtained by standard operations: cones, shifts, direct sums, and direct summands. G is called a *strong generator* if only finitely many such steps are needed. The minimal number of steps required to generate the entire category from G is its *generation time*. The collection of all generation times forms the *Orlov spectrum* of the category, and its infimum is the Rouquier dimension of $\mathcal{T}$. A *semiorthogonal decomposition* of $\mathcal{T}$ is a sequence of full subcategories arranged so that morphisms go only in one direction, and together they generate the whole category.

2.1. Orlov Spectra of Weighted Projective Lines

The Orlov spectrum was introduced by Orlov in [22], building on earlier work of Bondal–Van den Bergh and Rouquier. Since then, the study of spectra and the gaps between generation times has become an important theme in algebraic geometry and homological mirror symmetry, as developed for instance by Ballard, Favero, Katzarkov, and Takahashi [4, 31]. Despite its significance for understanding triangulated categories, the Orlov spectrum has been explicitly computed in only a few cases. One such case is that of the Dynkin quivers A_n. Since A_n is of finite representation type, every object is a direct sum of finitely many indecomposable modules. In [4], Ballard, Favero, and Katzarkov determined the Orlov spectrum of $\mathrm{D^b_{mod}}(kA_n)$, where A_n is the Dynkin quiver of type A given by

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n+1$$

Weighted projective lines $\mathbb{P}(a, b)$, when viewed as a stacky curve, forms a natural generalization of the projective line. Like the projective line, the derived category of $\mathbb{P}(a, b)$ can also be associated with the derived category of some quiver algebra. It was first introduced by Geigle and Lenzing in [16] and since then, it has found relevance in the fields of algebraic geometry and representation theory. These can be found in the works of Krause and Chen in [10], Ruan and Wang in [30], Dimitrov and Katzarkov in [12] [13], [24].

In this project with my advisor Matthew Ballard, we study the derived category of weighted projective lines $\mathbb{P}(1, 2)$ and explicitly compute its generation times and the Orlov spectrum. In this project, we compute the exact generation time for particular objects. Specifically, we have:

Lemma 2.1 (Ballard, B.) *Let G be a strong generator in $\mathrm{D^b_{coh}}(\mathbb{P}(1, 2))$. Then*

(1) $G = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$ has generation time 1.
(2) $G = \mathcal{O}(-1) \oplus T_{1,q} \oplus \mathcal{O}$ has generation time 2.
(3) $G = \mathcal{O} \oplus T_{2,q}(-1)$ has generation time 3.

We also give an upper bound on the generation time of any strong generator G in this category. These results help us to prove our main theorem:

Theorem 2.2 (Ballard, B.) *The Orlov spectrum of $\mathrm{D^b_{coh}}(\mathbb{P}(1, 2))$ is $\{1, 2, 3\}$.*

Similarly to the example 1.1, we have an association of $\mathrm{D^b_{coh}}(\mathbb{P}(1, 2))$ with $\mathrm{D^b_{mod}}(k\tilde{A}_3)$, where $\tilde{A}_3$.

Corollary 2.3 (Ballard, B.) *The Orlov spectrum of $\mathrm{D^b_{mod}}(k\tilde{A}_3)$ is $\{1, 2, 3\}$.*

Here, $\tilde{A}_3$ is the extended Dynkin quiver of type A:

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3 \end{tikzcd}$$

In our ongoing project, we are trying to make progress in the following conjecture:

Conjecture 2.4 (Ballard, B.) *The Orlov spectrum of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}(a,b))$ *is* $\{1, 2, \cdots, a+b\}$.

Proving this also sheds light on the derived categories of extended Dynkin quivers $\widetilde{A}_n$. For $\mathbb{P}(1,n)$, the methods used in $\mathbb{P}(1,2)$ generalize: identifying combinations of objects that fail to generate, and bounding generation time when they do. For my future research, I plan to pursue the following directions:

- Generation in derived categories of weighted projective lines of higher weights: I aim to study generation in the derived categories of weighted projective lines with higher weights by relating them to those of lower weights. This approach will build upon the techniques developed in [14], Proposition 6.13, and [11], Proposition 6.5.
- Tubular weighted projective lines: I intend to explore the derived category of tubular weighted projective lines through their relationship with Atiyah's classification of vector bundles on smooth elliptic curves and Ringel's classification of modules over canonical algebras of tubular type.
- Thick subcategories and generation times in hereditary abelian categories: More generally, I am interested in understanding the structure of derived categories of hereditary abelian categories with indecomposable objects. As a first step, this project will involve: (a) classifying thick subcategories of weighted projective lines $\mathbb{P}(a,b)$ following [11], and (b) for categories with well-understood indecomposables, such as $\mathbb{P}(1,n)$, estimating generation times by determining the lengths of maximal ghost sequences for each type of indecomposable. This work may also incorporate computational methods and programming assistance to handle explicit examples.

2.2. **Preservation for Generation**

One way of understanding dimension and generation in derived categories in the context of non-commutative algebraic geometry is to relate them to the derived categories of more familiar objects, such as commutative schemes.

Rouquier dimension provides a measure of the complexity of a derived category. Introduced by Rouquier in [29] as a lower bound for the representation dimension of finite-dimensional algebras, it also plays an important role in algebraic geometry: he proved that the Rouquier dimension of a smooth projective variety is bounded above by twice its Krull dimension, and Orlov conjectured in [22] that $\mathrm{Rdim}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) = \dim(X)$ whenever X is a smooth quasi-projective scheme.

In recent joint work with Lank and Dey [6], we studied non-commutative schemes $(X, \mathcal{A})$, where X is a scheme and $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra. Our goal was to relate generation in such non-commutative settings to that of the underlying commutative scheme, and to establish bounds on Rouquier dimension. We proved the following theorem:

Theorem 2.5 (B., Dey, Lank) *Let X be a Noetherian J-2 scheme of finite Krull dimension, and let $\pi\colon \mathcal{O}_X \to \mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra with full support. If G is a classical generator of $D^b_{\mathrm{coh}}(\mathcal{A})$, then $\mathbf{R}\pi_* G$ is a classical generator of $D^b_{\mathrm{coh}}(X)$.*

This shows that generators on the non-commutative side descend to the commutative side. As a consequence, we obtained a general lower bound:

Theorem 2.6 (B., Dey, Lank) *If X is integral, Jacobson, catenary, Noetherian, and J-2, and if $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra with full support, then*

$$\operatorname{Rdim}\left(D^b_{\text{coh}}(\mathcal{A})\right) \;\geq\; \dim(X).$$

These results connect the Rouquier dimension of non-commutative schemes to fundamental invariants of the underlying commutative variety, providing a categorical tool to measure their complexity.

2.3. **Dimension Theory of Non-commutative Curves**

Another fascinating way to understand and work with Non-commutative schemes is through a categorical point of view. In [28], Reiten and Van den Bergh gave a categorical notion of a non-commutative curve. They defined a non-commutative curve as follows:

Definition 2.7. A *non-commutative curve* is a $k-$linear Ext-finite abelian category with homological dimension one.

In [28], Theorem v.1.2. Reiten and Van den Bergh point out that if $\mathcal{A}$ is a non-commutative curve and it is Noetherian, smooth and connected, then $\mathcal{A}$ can be one of the two possibilities: either it can be the category of modules over the path algebra of an acyclic quiver or it can be the category of coherent sheaves on a smooth projective stacky curve. In [22], Theorem 6, Orlov shows that the Rouquier dimension of the bounded derived category of a smooth projective curve of genus ≥ 1 is one. In a categorical context, the significance of computing dimensions becomes more relevant in order to understand non-commutative schemes. It is therefore natural to ask the following question:

Question 2.8 *What is the Rouquier dimension of a non-commutative curve?*

In [13], Elagin and Lunts computed the Rouquier dimension for finite acyclic quivers (both ADE and non-ADE), as well as for orbifold projective lines. In their earlier work [29], Rouquier introduced additional invariants of triangulated categories, namely the Serre dimension and the diagonal dimension. Ballard and Favero [3] defined the diagonal dimension of a variety and established bounds for the diagonal dimension of a variety or a Deligne–Mumford stack. Later, Ikeda and Qiu [17] introduced the notion of the global dimension of stability conditions. The diagonal dimension of a smooth projective curve was determined in [21], while the global dimension of an orbifold projective line has been computed in [24]. In collaboration with Antonios–Alexandros Robotis and Isaac Goldberg, we aim to extend these results and address the following question:

Question 2.9 *What are the values of Rouquier dimension, diagonal dimension, Serre dimension and global dimension of a non-commutative curve?*

In this joint work, we have generalized the results on dimensions of derived categories of coherent sheaves on smooth projective varieties to smooth orbifold curves. These help us to complete the computations for dimensions of non-commutative curves.

Theorem 2.10 (B., Robotis, Goldberg) *Let $\mathcal{X}$ be an orbifold curve. Then:*

- *(a) the Rouquier dimension of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X})$ is one.*
- *(b)* $\operatorname{Ddim}(\mathcal{X}) = 1$ *if and only if* $\deg(\omega_{\mathcal{X}}) < 0$. *Otherwise,* $\operatorname{Ddim}(\mathcal{X}) = 2$.
- *(c) If $\mathcal{X}$ is a smooth and projective Deligne–Mumford stack over a field k, then* $\overline{\operatorname{Sdim}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})) = \underline{\operatorname{Sdim}}(\mathrm{D}^{\mathrm{b}}(\mathcal{X})) = n$.
- *(d) If $\mathcal{X}$ is an orbifold curve over $\mathbb{C}$, then* $\operatorname{gldim}(\mathcal{X}) = 1$.

This project adds to the study of dimensions for a non-commutative curve. This also shows that the conjecture by Elagin and Lunts in [15] holds true in the case of non-commutative curve. They conjectured that for a smooth and regular differential graded algebra A over a field such that the upper and lower Serre dimension of $\mathrm{perf}(A)$, the category of perfect complexes over A are equal, the following inequality holds: $\mathrm{Rdim}(\mathrm{perf}(A)) \leq \mathrm{Sdim}(\mathrm{perf}(A)) \leq \mathrm{Ddim}(\mathrm{perf}(A))$.

In this project, we have shown that the Rouquier dimension of a non-commutative curve is one. It would be interesting to know if the converse holds.

Question 2.11 *Let $\mathcal{A}$ be an abelian category such that $\mathrm{Rdim}\,\mathrm{D^b}(\mathcal{A}) \leq 1$. Then is $\mathcal{A}$ a non-commutative curve?*

Abelian categories $\mathcal{A}$ whose derived categories $\mathrm{D^b}(\mathcal{A})$ have Rouquier dimension zero have long been classified. In the future, I would like to work on a classification of $\mathcal{A}$ for which $\mathrm{Rdim}\,\mathrm{D^b}(\mathcal{A}) = 1$. I would like to know for which other abelian categories the inequality conjectured by Elagin and lunts hold true. One approach to address this conjecture, in the case of certain DG-algebras(like Calabi-Yau DG algebras) would be to associate t-structures with generation.

Non-commutative schemes, as described by Orlov in [23], can also be studied from the point of view of differential graded algebras over a field. This makes the study of DG-algebras more interesting. In the future, similar to the motivation of the project 2.2, I plan to study the properties of the bounded derived category of perfect complexes on a DG-algebra A and relate properties like strong generation, t-structures, silting objects with the bounded derived category of modules over the $k-$algebra $H^0(A)$, the zeroth homology of A, since finite dimensional algebras and their derived categories are comparatively well known. These works are in the spirit of works by Stevenson, Brown, Shridhar, Levins, Elagin, Lunts, and Orlov in [9, 15, 23, 27].

2.4. **Derived McKay Correspondence and Semi-Orthogonal Decomposition**

Similar to the notions of generation and dimension, the study of larger derived categories often requires breaking them down into more manageable pieces. A key tool for this is semiorthogonal decomposition, which allows us to understand a derived category by analyzing its smaller components that together generate the whole.

The Derived McKay Correspondence Conjecture states that for a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$, the bounded derived category of coherent sheaves on any crepant resolution Y of $\mathbb{C}^n/G$ is equivalent to the derived category of finitely generated modules over the skew group algebra [18, 19]. In joint work originating from the AMS MRC 2023, we studied reflection groups of rank two generated by order-two reflections. Building on the conjecture of Polishchuk–Van den Bergh [25], which predicts that the components of the semi-orthogonal decomposition of $D^G(X)$ correspond to the irreducible representations of G, we proved the following:

Theorem 2.12 (B., Davidov, Faber, Honigs, McDonald, Overton-Walker, Spence, [5] Theorem A) *Let $G = G(2m, m, 2)$ with $m \geq 3$, or G_{12}, G_{13}, G_{22}. Then*

$$D^G(\mathbb{C}^2) \cong \langle D(B_1), \ldots, D(B_r), E_1, \ldots, E_n, D(\mathbb{C}^2/G)\rangle,$$

where B_i are the normalizations of the irreducible components of the branch divisor, E_j are exceptional objects, and $r + n + 1$ equals the number of irreducible representations of G.

Together with Potter's work [26], this confirms the Polishchuk–Van den Bergh conjecture for groups $G \leq \mathrm{GL}(2,\mathbb{C})$ generated by order-two reflections, via explicit computations on the H-Hilbert scheme.

References

[1] Michael Artin and James J Zhang, *Noncommutative projective schemes*, Advances in mathematics **109** (1994), no. 2, 228–287.

[2] Dagmar Baer, *Tilting sheaves in representation theory of algebras*, Manuscripta mathematica **60** (1988), 323–347.

[3] Matthew Ballard and David Favero, *Hochschild dimensions of tilting objects*, International Mathematics Research Notices **2012** (2012), no. 11, 2607–2645.

[4] Matthew Ballard, David Favero, and Ludmil Katzarkov, *Orlov spectra: bounds and gaps*, Inventiones mathematicae **189** (2012), no. 2, 359–430.

[5] Anirban Bhaduri, Yael Davidov, Eleonore Faber, Katrina Honigs, Peter McDonald, C Eric Overton-Walker, and Dylan Spence, *An explicit derived mckay correspondence for some complex reflection groups of rank two*, arXiv preprint arXiv:2412.17937 (2024).

[6] Anirban Bhaduri, Souvik Dey, and Pat Lank, *Preservation for generation along the structure morphism of coherent algebras over a scheme*, Bulletin of the London Mathematical Society (2025).

[7] A Bondal, *Helices, representations of quivers and koszul algebras. helices and vector bundles, 75–95*, London Math. Soc. Lecture Note Ser **148**.

[8] Alexei Bondal and Dmitri Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Mathematica **125** (2001), no. 3, 327–344.

[9] Michael K Brown, Andrew J Soto Levins, and Prashanth Sridhar, *Existence of balanced dualizing dg-modules*, arXiv preprint arXiv:2506.02398 (2025).

[10] Xiao-Wu Chen and Henning Krause, *Introduction to coherent sheaves on weighted projective lines*, arXiv preprint arXiv:0911.4473 (2009).

[11] Yiyu Cheng, *Wide subcategories of a domestic weighted projective line*, Journal of Pure and Applied Algebra **228** (2024), no. 9, 107669.

[12] George Dimitrov and Ludmil Katzarkov, *Bridgeland stability conditions on wild kronecker quivers*, Advances in Mathematics **352** (2019), 27–55.

[13] Alexey Elagin, *Calculating dimension of triangulated categories: path algebras, their tensor powers and orbifold projective lines*, Journal of Algebra **592** (2022), 357–401.

[14] ———, *Thick subcategories on weighted projective curves and nilpotent representations of quivers*, arXiv preprint arXiv:2407.01207 (2024).

[15] Alexey Elagin and Valery A Lunts, *Three notions of dimension for triangulated categories*, Journal of Algebra **569** (2021), 334–376.

[16] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, Singularities, representation of algebras, and vector bundles: Proceedings of a symposium held in lambrecht/pfalz, fed. rep. of germany, dec. 13–17, 1985, 2006, pp. 265–297.

[17] Akishi Ikeda and Yu Qiu, *q-stability conditions on calabi-yau-x categories*, arXiv preprint arXiv:1807.00469 (2018).

[18] Akira Ishii, *On the mckay correspondence for a finite small subgroup of gl (2,)* Journal für die reine und angewandte Mathematik (Crelles Journal) **2002** (2002), no. 549, 221.

[19] Mikhail Kapranov and Eric Vasserot, *Kleinian singularities, derived categories and hall algebras*, arXiv preprint math (1998).

[20] Henning Krause, *Homological theory of representations*, Vol. 195, Cambridge University Press, 2021.

[21] Noah Olander, *Diagonal dimension of curves*, International Mathematics Research Notices **2024** (2024), no. 10, 8172–8184.

[22] Dmitri Orlov, *Remarks on generators and dimensions of triangulated categories*, arXiv preprint arXiv:0804.1163 (2008).

[23] ———, *Smooth and proper noncommutative schemes and gluing of dg categories*, Advances in Mathematics **302** (2016), 59–105.

[24] Takumi Otani, *Global dimension of the derived category of an orbifold projective line*, arXiv preprint arXiv:2306.16673 (2023).

[25] Alexander Polishchuk and Michel Van den Bergh, *Semiorthogonal decompositions of the categories of equivariant coherent sheaves for some reflection groups.*, Journal of the European Mathematical Society (EMS Publishing) **21** (2019), no. 9.

[26] Rory Potter, *Derived categories of surfaces and group actions.*, Ph.D. Thesis, 2017.

[27] Theo Raedschelders and Greg Stevenson, *Proper connective differential graded algebras and their geometric realizations*, European Journal of Mathematics **8** (2022), no. Suppl 2, 574–598.

[28] Idun Reiten and Michel Van den Bergh, *Noetherian hereditary abelian categories satisfying serre duality*, Journal of the American Mathematical society **15** (2002), no. 2, 295–366.

[29] Raphaël Rouquier, *Dimensions of triangulated categories*, Journal of K-theory **1** (2008), no. 2, 193–256.

[30] Shiquan Ruan and Xintian Wang, *t-stabilities for a weighted projective line*, Mathematische Zeitschrift **297** (2021), no. 3, 1119–1160.

[31] Ryo Takahashi, *Uniformly dominant local rings and orlov spectra of singularity categories*, arXiv preprint arXiv:2412.00669 (2024).